Parameter Estimation for Wavelet Transformed Ultrasonic Signals

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Motivation

- Wavelet transform is an integration procedure which has applications in:
  - Signal and Image Processing
  - Wave Analysis
  - Material Evaluation
  - Non-Destructive Testing and Evaluation
- Develop an algorithm to estimate the parameters of wavelet transformed ultrasonic signals.

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Wavelet Transform

The continuous wavelet transform of a signal $x(t)$ is defined in a convolution form as

$$W(a,b) = \int x(t)\bar{\psi}_{a,b}(t)dt,$$

where $\bar{\psi}_{a,b}(t)$ is the complex conjugate wavelet function, a is the scaling factor, and b is the translation factor. $\psi_{a,b}(t)$ has the form of

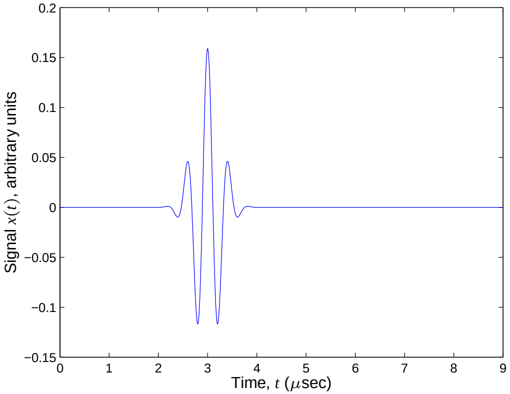
$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right),$$

where $\psi(t_w)$ is the wavelet function and $t_w = \frac{t-b}{a}$.

Gaussian Spectrum Signal

Consider a Gaussian spectrum signal shifted in time by t_c

$$x(t-t_c) = \frac{1}{2\pi} e^{-j\omega_c (t-t_c)} e^{-\frac{1}{8}B^2(t-t_c)^2},$$



Wavelet Transform of Gaussian Spectrum Signal

Using the Morlet wavelet function

$$\psi(t_w) = e^{-j2\pi t_w} e^{-\frac{1}{2}t_w^2},$$

the closed-form solution of the wavelet transform of the Gaussian spectrum signal $x(t - t_c)$ is

$$W(\tau, \omega; t_c) = A(\omega) e^{\kappa(\tau - t_c, \omega)} e^{j\chi(\tau - t_c, \omega)},$$

where

$$A(\omega) = \frac{1}{2}\sqrt{\frac{2\eta}{\pi(B^2 + 4\eta^2)}} e^{\frac{-2(\omega-\omega_c)^2}{B^2+4\eta^2}},$$

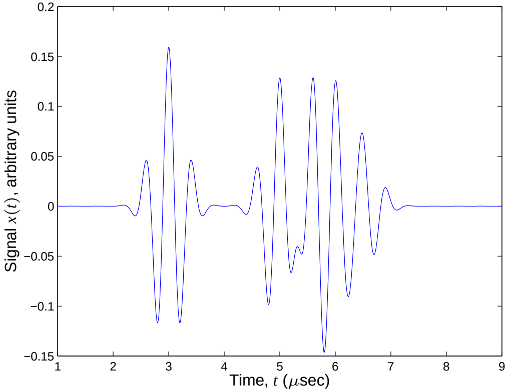
$$\kappa(\tau - t_c, \omega) = \frac{-B^2\eta^2}{2(B^2 + 4\eta^2)} (\tau - t_c)^2,$$

$$\chi(\tau - t_c, \omega) = \frac{4\omega_c\eta^2 (t_c - \tau) + B^2\omega (t_c - \tau)}{B^2 + 4\eta^2}.$$

Gaussian Spectrum Signal with Multiple Echoes

Consider a Gaussian spectrum signal with N reflected echoes

$$x(t - t_c^i; \{c_i\}, \{t_c^i\}) = \sum_{i=1}^{N} \frac{c_i}{2\pi} e^{-j\omega_c (t - t_c^i)} e^{-\frac{1}{8}B^2 (t - t_c^i)^2},$$



Wavelet Transform of Multiple Echoes

From the superposition principle, the closed-form analytical solution of the wavelet transform is

$$\tilde{W}(\tau,\omega;\{c_i\},\{t_c^i\}) = \sum_{i=1}^{N} c_i A(\omega) e^{\kappa(\tau - t_c^i,\omega)} e^{j\chi(\tau - t_c^i,\omega)},$$

where

$$A(\omega) = \frac{1}{2}\sqrt{\frac{2\eta}{\pi(B^2+4\eta^2)}}\, e^{\frac{-2(\omega-\omega_c)^2}{B^2+4\eta^2}},$$

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Goal: Recover $\theta^\star := [c_1, \cdots, c_N, t_c^1, \cdots, t_c^N]^T \in \mathbb{R}^{2N}$

Nonlinear Least Squares Estimate

The nonlinear least squares (NLS) estimate $\hat{\theta}$ is found by

$$\min_{\theta \in \mathbb{R}^{2N}} S(\theta), \quad S(\theta) := \sum_{\tau_m=1}^{\tau_M} \sum_{\omega_m=1}^{\omega_M} \left| y(\tau,\omega) - \tilde{W}(\tau,\omega;\theta) \right|^2$$

where $y(\tau,\omega)$ represents the noisy version of the closed-form model $\tilde{W}(\tau,\omega;\theta^\star)$

$$y(\tau,\omega) = \tilde{W}(\tau,\omega;\theta^\star) + v(\tau,\omega),$$

and $v \sim \mathcal{N}(0,\sigma_v^2)$ is the Gaussian white noise due to the measurement and numerical errors.

Estimate Covariance Matrix

For M number of post-processed data points the covariance matrix P of the estimate $\hat{\theta}$ is obtained by

$$P := \mathbb{E}(\hat{\theta} - \mathbb{E}(\hat{\theta}))(\hat{\theta} - \mathbb{E}(\hat{\theta}))^T.$$

From the Cramer-Rao theorem, the covariance matrix P of any unbiased estimator is lower bounded by the Cramer-Rao Lower Bound (CRLB), or the inverse of the Fisher Information Matrix (FIM)

$$P \succeq \mathrm{FIM}(\theta^\star)^{-1} = \frac{\sigma^2 \Sigma_{2M}^{-1}(\theta^\star)}{M} =: \mathrm{CRLB}$$

where

$$\Sigma_{2M} := \frac{1}{M} \sum_{\tau_m=1}^{\tau_M} \sum_{\omega_m=1}^{\omega_M} \mathrm{Re}\left\{\tilde{W}'(\tau,\omega;\theta^\star)\tilde{W}'(\tau,\omega;\theta^\star)^*\right\}.$$

Separable Nonlinear Least Squares

Since the c_i parameters appear linearly, the residual $S(\theta)$ can be expressed as

$$S(\theta) = \sum_{\tau_m=1}^{\tau_M} \sum_{\omega_m=1}^{\omega_M} |y(\tau,\omega) - h(\tau,\omega)\theta_c|^2$$

where

$$\begin{aligned} \theta_c &= [c_1 \quad c_2 \quad \cdots \quad c_N]^{\mathrm{T}}, \\ h(\tau,\omega) &= [h_1(\tau,\omega) \quad h_2(\tau,\omega) \quad \cdots \quad h_N(\tau,\omega)], \end{aligned}$$

and

$$h_i(\tau,\omega) = A(\omega) e^{\kappa(\tau - t_c^i,\omega)} e^{j\chi(\tau - t_c^i,\omega)}.$$

Separable Nonlinear Least Squares

The residual $S(\theta)$ can be re-written as

$$S(\theta_c, \theta_{t_c}) = \left(Y - H(\theta_{t_c})\theta_c\right)^{\mathrm{T}} \left(Y - H(\theta_{t_c})\theta_c\right),$$

where

$$Y = \begin{bmatrix} y(\tau_1, \omega_1) \\ \vdots \\ y(\tau_1, \omega_M) \\ \vdots \\ y(\tau_M, \omega_1) \\ \vdots \\ y(\tau_M, \omega_M) \end{bmatrix} \quad \text{and} \quad H(\theta_{t_c}) = \begin{bmatrix} h(\tau_1, \omega_1) \\ \vdots \\ h(\tau_1, \omega_M) \\ \vdots \\ h(\tau_M, \omega_1) \\ \vdots \\ h(\tau_M, \omega_M) \end{bmatrix},$$

and

$$\theta_{t_c} = \begin{bmatrix} t_c^1 & t_c^2 & \cdots & t_c^N \end{bmatrix}^{\mathrm{T}}.$$

Separable Nonlinear Least Squares

For a given θ_{t_c}, the θ_c that minimizes $S(\theta_c, \theta_{t_c})$ is the linear least squares estimate

$$\hat{\theta}_c = \left(H^{\mathrm{T}}(\theta_{t_c})H(\theta_{t_c})\right)^{-1} H^{\mathrm{T}}(\theta_{t_c})Y.$$

When $\hat{\theta}_c$ is substituted into $S(\theta_c, \theta_{t_c})$ the residual is

$$S(\hat{\theta}_c, \theta_{t_c}) = Y^{\mathrm{T}} \left[I - H(\theta_{t_c}) \left(H^{\mathrm{T}}(\theta_{t_c})H(\theta_{t_c})\right)^{-1} H^{\mathrm{T}}(\theta_{t_c}) \right] Y,$$

which reduces the estimation of θ_{t_c} to the maximization of

$$\max_{\theta_{t_c} \in \Theta_{t_c}} F(\theta_{t_c}); \quad F(\theta_{t_c}) = Y^{\mathrm{T}} H(\theta_{t_c}) \left(H^{\mathrm{T}}(\theta_{t_c})H(\theta_{t_c})\right)^{-1} H^{\mathrm{T}}(\theta_{t_c})Y,$$

where $\Theta_{t_c} = \{\theta_{t_c} : \tau_{\mathrm{min}} \leq \theta_{t_c} \leq \tau_{\mathrm{max}}\}$.

Separable Nonlinear Least Squares

The initial conditions for the maximization of $F(\theta_{t_c})$ are obtained by a grid search of

$$\bar{F}(t_c) = Y^{\mathrm{T}}\bar{H}(t_c)\left(\bar{H}^{\mathrm{T}}(t_c)\bar{H}(t_c)\right)^{-1}\bar{H}^{\mathrm{T}}(t_c)Y$$

over $\tau_{\min} \leq t_c \leq \tau_{\max}$, with

$$\bar{H}(t_c) = \begin{bmatrix} \bar{h}(\tau_1,\omega_1) \\ \vdots \\ \bar{h}(\tau_1,\omega_M) \\ \vdots \\ \bar{h}(\tau_M,\omega_1) \\ \vdots \\ \bar{h}(\tau_M,\omega_M) \end{bmatrix},$$

and $\bar{h}(\tau,\omega) = A(\omega)e^{\kappa(\tau - t_c,\omega)}e^{j\chi(\tau - t_c,\omega)}$.

Peaks form near the t_c locations for the measured data.

Optimal Number of Echoes

The Akaike Information Criterion (AIC) is given by

$$AIC = \underbrace{M \ln\left(\frac{S}{M}\right)}_{AIC_1} + \underbrace{2\left(2N+1\right)}_{AIC_2}$$

where S, N, and M are, respectively, the residual, the number of echoes, and the sample size.

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Algorithm for Separable Nonlinear Least Squares

Input: Gaussian spectrum signal $x(t)$, the center frequency ω_c, and the bandwidth B.

Output: The estimated parameters $\hat{\theta}$ and the number of echoes N.

- Compute wavelet transform coefficients of $x(t)$.
- Determine lower and upper bounds on the number of echoes $N \in [N_L, N_U]$.
- **for** $N = N_L{:}N_U$ **do**
  - The t_c locations of the N highest peaks of $\bar{F}(t_c)$ are recorded into t_{c0}.
  - Starting at t_{c0}, perform the maximization of $F(\theta_{t_c})$ to find the estimated time of flights $\hat{\theta}_{t_c}$.
  - Using the estimated time of flights $\hat{\theta}_{t_c}$, compute $\hat{\theta}_c$.
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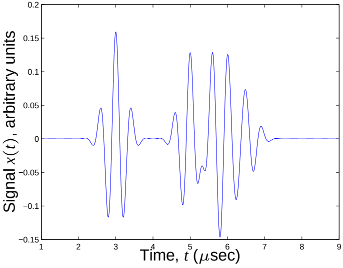
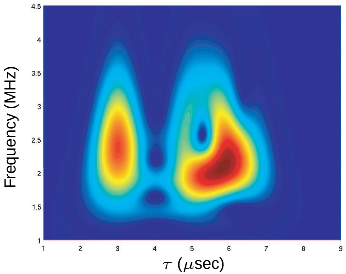
Realistic Multiple Echoes



(a)



(b)

Table: Comparison of the true parameters to the SNLS estimates. The values for t_c^i are expressed in microseconds.

	c_1	c_2	c_3	c_4	c_5	t_c^1	t_c^2	t_c^3	t_c^4	t_c^5
$\theta^\star$	1.0000	0.8500	0.7000	0.5500	0.4000	3.0000	5.0000	5.6000	6.0000	6.5000
$\hat{\theta}$	1.0000	0.8500	0.6999	0.5500	0.4000	2.9999	5.0000	5.5999	5.9999	6.4999

Noisy Observations

A Monte-Carlo (MC) simulation was performed with 5000 realizations at a noise level of $\sigma_v = \frac{0.1\max(\tilde{W})}{3}$, such that the maximum value of the noise is no more than 10% of the maximum of $\tilde{W}$.

	$\theta^\star$	MC mean	MC Variance	CRLB Variance
c_1	10.0	9.9976	0.0042	0.0042
c_2	7.0	6.9955	0.0042	0.0042
c_3	6.0	5.9944	0.0043	0.0042
t_c^1	2.5	2.5000	0.5398×10^{-18}	4.0929×10^{-25}
t_c^2	3.0	3.0000	1.0994×10^{-18}	7.5104×10^{-25}
t_c^3	4.0	4.0000	1.5329×10^{-18}	5.6529×10^{-25}

Questions?